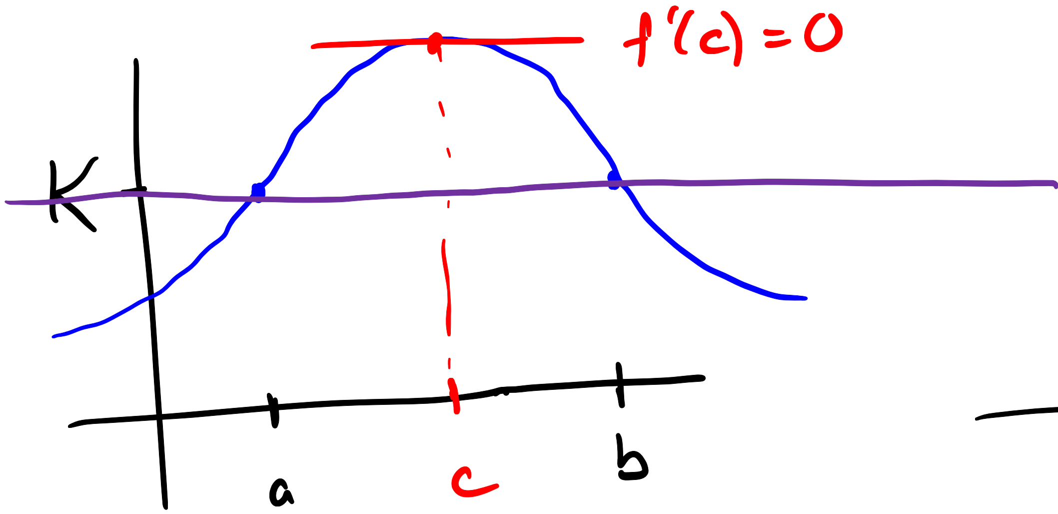
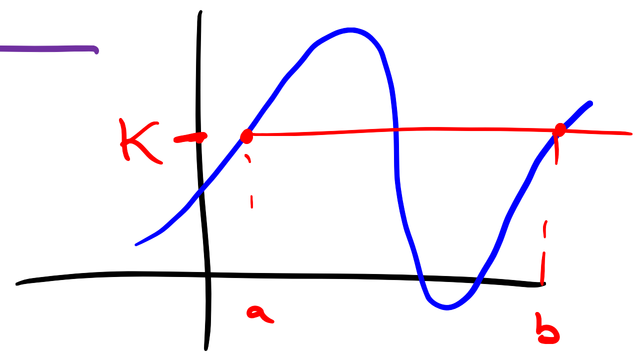
Rolle's Theorem

Let f be a function satisfying:

1. f is continuous on $[a,b]$
2. f is differentiable on (a,b)
3. $f(a) = f(b)$

Then there is a number c in (a,b) such that $f'(c)=0$





Proof:

(trivial case)

Case 1: $f(x)=k$ (constant function)

Then $f'(x)=0$ for all numbers in (a,b). So, c can be any number in (a,b).

<u>Case 2</u>: $f(x) > f(a)$ for some x in (a,b).

By the Extreme Value Theorem, f has an absolute maximum somewhere on $[a,b]$. Since $f(a) = f(b)$ and $f(x) > f(a)$ somewhere in (a,b), the abs. max. cannot be at a or b, so must occur at some number c in (a,b). Thus f has a local maximum at c, hence $f'(c) = 0$ because f is differentiable on (a,b).

<u>Case 3</u>: $f(x) < f(a)$ for some x in (a,b)

Similar to case 2, but use absolute minimum instead.

Ex: Prove that the equation $x^3+x-1=0$ has exactly one real root.

Proof: Let $f(x)=x^3+x-1$. Then $f(-1)=-3$ and $f(1)=1$, so by the intermediate value theorem, there is a number c in $(-1,1)$ such that $f(c)=0$.

Assume $f(x)$ has more than one root, i.e., $f(c_1)=0$ & $f(c_2)=0$ & $c_1<c_2$. Since $f(x)$ is differentiable on $(-\infty,\infty)$, by Rolle's Theorem, $f'(k)=0$ for some k in (c_1,c_2). But $f'(x)=3x^2+1\geq 1$, so $f'(k)$ cannot be zero. Therefore f has only one real root. D

The Mean Value Theorem

Let f be a function satisfying:

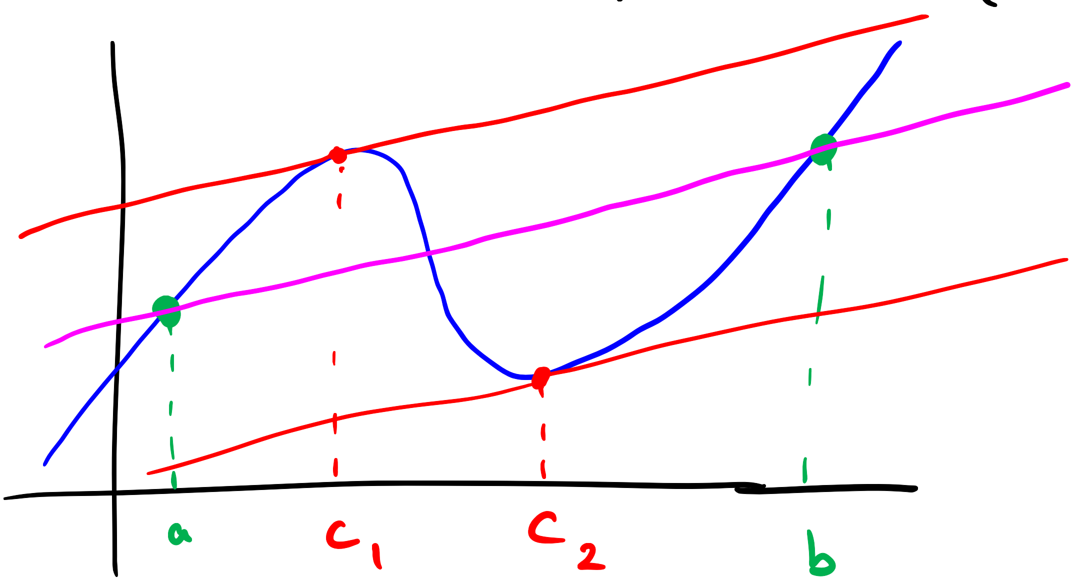
1. f is continuous on $[a,b]$
2. f is differentiable on (a,b)

Then, there is a number c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

or, equivalently,

$$f(b)-f(a) = f'(c)\,(b-a)$$



$$m_{ab} = \frac{f(b)-f(a)}{b-a}$$

Ex: At 2pm a car's speedometer reads 30 mi/hr. At 2:10pm it reads 50 mi/hr. Show at some point between 2pm and 2:10pm the car's acceleration is exactly 120 mi/hr^2.

Sol: $f(t)$ = car's speed

$t=0$ is 2pm & $t=\frac{1}{6}$ is 2:10pm

then, $$f'(c) = \frac{f(\frac{1}{6})-f(0)}{\frac{1}{6}-0} = \frac{50-30}{\frac{1}{6}} = 120$$